Measuring microwave quantum states: tomogram and moments

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Two measurable characteristics of microwave one-mode photon states are discussed: a rotated quadrature distribution (tomogram) and normally/antinormally ordered moments of photon creation and annihilation operators. Extraction of these characteristics from amplified microwave signal is presented. Relations between the tomogram and the moments are found and can be used as a cross check of experiments. Formalism of the ordered moments is developed. The state purity and generalized uncertainty relations are considered in terms of moments. Unitary and non-unitary time evolution of moments is obtained in the form of a system of linear differential equations in contrast to partial differential equations for quasidistributions. Time evolution is specified for the cases of a harmonic oscillator and a damped harmonic oscillator which describe noiseless and decoherence processes, respectively.

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I. INTRODUCTION

Non-classical states of light are of great interest not only from the viewpoint of academic science but also from the viewpoint of their promising quantum information applications such as cryptography, communication, etc. Any quantum experiment consists of the following procedures: preparation of a desired quantum state, time evolution of the state (its transformation through a quantum channel), and a measurement. The latter one is a keystone to get an insight into quantum world. The result of a measurement is usually a set of outcomes distributed in accordance with the quantum nature of the state. Given a numerous number of identically prepared quantum states and neglecting the memory effects of channels, one can perform many individual measurements and gather the statistical information. We refer to the obtained information as a measurable characteristic of the state. At this stage one encounters a problem how a measurable characteristic is related with the quantum state itself, i.e. how to represent the results of measurements.

A quantum state is usually described by wave function or density operator [1]. Alternatively, different representations of the density operator are widely used, e.g., the Wigner function [2]. Recently, a probability representation of quantum states was introduced [3]. In the probability representation, quantum states are identified with fair probability distributions. The probability representation was introduced in connection with homodyne measuring the photon quantum states by means of optical tomography. Using the homodyne detection scheme gives rise to the optical tomogram $w(X,\theta)$ which has a meaning of the probability distribution of a single rotated quadrature component $\hat{X}_\theta = \hat{q}\cos\theta + \hat{p}\sin\theta$ in the phase space of one light mode (see, e.g., the review [4]). The aim of homodyne measurements [5] was to obtain the Wigner function identified with the quantum state. Then the reconstructed Wigner function was used to extract an information on the physical properties. The measurements of this kind in optical frequency domain were fulfilled, e.g., in [6, 7], and the detailed review of the experimental results is presented in Ref. [8]. Though optical tomogram is a fair probability distribution, it was interpreted in all the experimental works as a technical ingredient providing tool to obtain the Wigner function. The latter one was interpreted as a 'real state'. On the contrary, following the ideas of the probability representation [3, 9] (see also the recent review [10] and the paper [11]), the optical tomogram and other kinds of tomographic-probability distributions like symplectic tomogram [12] were considered as a primary object containing complete information on quantum state. In view of this fact, one does not need reconstructing any quasidistribution including the Wigner function. The reconstruction procedure produces extra inaccuracies related to the useless elaborating the experimental data by means of Radon integral transforms. The experiment on direct checking purity-dependent uncertainty relations [13, 14] and measuring the photon state purity and temperature, without reconstruction of the Wigner function or another quasidistribution, was performed recently in Ref. [15]. In this experiment, the optical tomogram was measured and considered as a primary object determining the quantum state. Moreover, the above physical characteristics were expressed in terms of the tomogram, which is nothing else but directly measurable alternative to the density operator and the Wigner function.

On the other hand, at optical frequencies there is another measurable characteristic of the state. Indeed, using the heterodyne detection scheme gives rise to the Husimi function $Q(q,p)$ which also contains the full information about the quantum state [4]. Since both tomogram and Husimi function are extracted from experimental data, the relation between them (see, e.g., [16])

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can be considered as a cross-check of the experiment accuracy. Closely related to the Husimi function are ordered moments $\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle$ and $\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle$, where $\hat{a}^\dagger$ and $\hat{a}$ are photon creation and annihilation operators, respectively. In fact, the moments also contain the complete information on the state (see, e.g., [17–20]). An optical scheme for measuring moments is proposed, e.g., in [21]. The moments are complex functions of integers. The measurements give Husimi function-type characteristics corresponding to a two-mode state, from which the moments for one-mode state are calculated. In this sense, the method of moments is not completely direct measurement in contrast to the optical tomography. Also, the moments are not a probability distribution as the tomogram is. Nevertheless, if the moments are known, then all desired characteristics of the quantum state can be calculated.

The importance of microwaves in up-to-date quantum technologies can scarcely be overestimated. Using microwave wavelengths transforms the scale of experiments as compared to standard optical ones. As a result, at microwaves the highest-quality superconducting resonators [22] are built and successfully used in the microwave version of cavity quantum electrodynamics. Being applied in one-dimensional resonators together with superconducting qubits, microwaves have opened an opportunity to achieve the strongest ever coupling between the electromagnetic field and an artificial atom (qubit) within compact and integrable electrical circuits [23]. Experimentally realized couplings of microwaves to transmons [24] and mechanical oscillators [25] made them a really significant tool for further progress of quantum information.

On the other hand, microwave quantum states are of great interest per se as carriers of quantum information. However, measuring quantum state of electromagnetic field changes dramatically and becomes a challenge when microwave radiation is under investigation. Detection of microwave field (especially itinerant modes) is complicated by the low efficiency of photodetectors, although suggestions of high-efficiency microwave photon detectors are made [26, 27]. As a result, it is unlikely to carry out the photon counting measurements. In order to register the radiation reliably, amplifiers are widely used [28, 29] though they unavoidably add some noise. In addition, interferometry experiments in optics extensively utilize beam splitters, whereas in microwave engineering the role of a beam splitter for a single mode is played by hybrid junctions or power dividers [30, 31]. Similarly, a microwave signal is mixed with a local oscillator microwave field (via a nonlinear device called mixer) instead of passing a signal mode and a local oscillator mode through a beam splitter in optics.

In spite of the challenges dealt with microwave radiation, the measurements of the 'optical' tomogram and ordered moments are feasible and were reported recently [32–35]. Really, using a homodyne detection scheme with phase sensitive amplifiers less noisy than high-electron mobility transistors (e.g., a Josephson parametric amplifier [36]) enables, in principle, measuring tomogram $\tilde{w}(X,\theta)$. This experimentally accessible distribution $\tilde{w}(X,\theta)$ differs from the genuine tomogram of the state $w(X,\theta)$ because of the noise presented. However, the information of the probed state can still be extracted from the data. An analogue of the 8-port homodyne detection of optical photons is realized at microwave level by phase-insensitive amplifiers and an inphase quadrature (I/Q) mixer. The I/Q mixer provides two outputs q and p described by a single envelope function $S = q + ip$. Statistics of experimentally measurable quantities $q = \mathrm{Re}S$ and $p = \mathrm{Im}S$ allows constructing the histogram $\tilde{Q}(q,p)$. If there were no extra noise added, this quasi-probability distribution $\tilde{Q}(q,p)$ would be an appropriately scaled Husimi function $Q(q,p)$ of the quantum state. Averaging the Husimi function with complex function $(S^*)^l S^k = (q-ip)^l(q+ip)^k$ results in the mean value of anti-normally ordered operator (see, e.g., [37]), namely, $\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle$.

The aim of our paper is to consider both methods of measuring microwave quantum states, viz., the homodyne tomography and the measurement of ordered moments. We briefly discuss how to extract moments from data corresponding to the amplified signal. Using different experimental schemes, an access to normally and antinormally ordered moments can be realized. The relations between the tomogram and the moments are derived, which in case of 'optical' tomogram and normally ordered moments coincide with relations found in Refs. [38, 39]. This connections can be utilized to perform cross-check of measurements. We also suggest a test in the form of inequalities for experimentally measured moments (both quadrature and creation-annihilation ones). This test is equivalent to checking uncertainty relations but avoids unnecessary reconstruction of the Wigner function or density operator. Finally, we cannot help considering time evolution of the quantum state in terms of measurable characteristics. Time evolution in the tomographic probability representation was considered in Refs. [3, 10, 40]. In this paper, we fill this gap for moments. Namely, the Moyal equation [41] for the Wigner function is rewritten in terms of moments as well as an eigenstate problem of Hamiltonian is formulated and non-unitary evolution of damped electromagnetic field oscillator is considered for a partial case of generic study of the quantum oscillator with dissipation [42]. The latter problem is instructive to clarify decoherence phenomena in microwave experiments, where the decoherence occurs due to a finite conductivity of waveguide walls or a lossy dielectric.

The paper is organized as follows.

In Sec. II, normally and antinormally ordered moments are extracted from amplified signals and calculated for examples of Fock, coherent, even/odd coherent [43], and thermal states. In Sec. III a brief review of symplectic and optical tomograms is given. In Sec. IV, relations between the tomogram and the moments are derived as

well as purity is expressed in terms of moments. In Sec. V, inequalities for moments and generalized purity-dependent uncertainty relations are suggested in connection with experiments such as that in Refs. [35, 44, 45], where a deterministic generation of microwave quantum states is demonstrated. In Sec. VI, a unitary time evolution and a problem of Hamiltonian eigenstates are formulated for moments. In Sec. VII, a particular case of the damped time evolution of moments is analyzed. Finally, in Sec. VIII, conclusions and prospects are given.

II. MOMENTS

To anticipate, let us outline a linear parametric amplifier as a constituent of microwave engineering. The amplifier changes the quantum state of the signal and idler incident modes by an $\mathrm{SU}(1,1)$ transformation [46], i.e. the Bogoliubov transformation, which has the following form in the Heisenberg picture (see, e.g., [47, 48]):

$$\hat{b}_{\mathrm{s}} = \sqrt{g}\hat{a} + \sqrt{g-1}\hat{h}^{\dagger}, \tag{1}$$

$$\hat{b}_{\mathrm{i}} = \sqrt{g-1}\hat{a}^{\dagger} + \sqrt{g}\hat{h}, \tag{2}$$

where $\hat{b}_{\mathrm{s}}$ and $\hat{b}_{\mathrm{i}}$ are annihilation operators of the amplified signal and idler modes, respectively, $\hat{a}$ and $\hat{h}$ are annihilation operators of the original microwave signal and extra noise modes, respectively. Depending on the operation of the amplifier, the output mode can be described by either Eq. (1) or Eq. (2).

If the experiment exploits a heterodyne detection scheme [33–35], the measured data represent nothing else but an appropriately scaled histogram of the Husimi function $H_{\mathrm{amp}}(q,p)$, which differs from the Husimi function of the original signal $H(q,p)$. However, the information about the quantum state can be revealed by means of moments. In fact, averaging $H_{\mathrm{amp}}(q,p)$ with complex function $(S^*)^l S^k = (q-ip)^l (q+ip)^k$ results in the mean value of anti-normally ordered operator $\hat{b}^k(\hat{b}^{\dagger})^l$ [49], where $\hat{b}$ is determined by either (1) or (2). For the sake of brevity, we will consider the case $\hat{b}=\hat{b}_{\mathrm{s}}$ only, which is readily rewritten for the opposite case $\hat{b}=\hat{b}_{\mathrm{i}}$ if needed.

Thus, experimentally accessible moments read

$$\langle \hat{b}_{\mathrm{s}}^k (\hat{b}_{\mathrm{s}}^{\dagger})^l \rangle = \sum_{i=0}^{k}\sum_{j=0}^{l} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} l \\ j \end{pmatrix} g^{(i+j)/2} \times (g-1)^{(k+l-i-j)/2} \langle \hat{a}^i (\hat{a}^{\dagger})^j \rangle \langle (\hat{h}^{\dagger})^{k-i} \hat{h}^{l-j} \rangle, \tag{3}$$

where the signal mode a and the noise mode h are assumed to be independent. Once gain g is known, average value of the normally-ordered noise operators $\langle (\hat{h}^{\dagger})^m \hat{h}^n \rangle$, $m,n=0,1,\ldots$ can be measured by analyzing vacuum state $|0\rangle$ of mode a for which $\langle 0|\hat{a}^i(\hat{a}^{\dagger})^j|0\rangle = i!\delta_{ij}$. Substituting this result in (3), we obtain an infinite system of linear equations of the form

$$\langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^k (\hat{b}_{\mathrm{s}}^{\dagger})^l|\mathrm{vac}\rangle = \sum_{i=0}^{\min(k,l)} \frac{k!l!}{i!(k-i)!(l-i)!} \times g^i (g-1)^{(k+l)/2-i} \langle (\hat{h}^{\dagger})^{k-i}\hat{h}^{l-i}\rangle. \tag{4}$$

Solution of this system is readily found for lower moments. If $n=0$ or $m=0$, we obtain

$$\langle (\hat{h}^{\dagger})^0 \hat{h}^n \rangle = (g-1)^{-n/2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^0 (\hat{b}_{\mathrm{s}}^{\dagger})^n|\mathrm{vac}\rangle, \tag{5}$$

$$\langle (\hat{h}^{\dagger})^m \hat{h}^0 \rangle = (g-1)^{-m/2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^m (\hat{b}_{\mathrm{s}}^{\dagger})^0|\mathrm{vac}\rangle. \tag{6}$$

Similarly, the cases $n=1$ or $m=1$ yield

$$\langle (\hat{h}^{\dagger})^1 \hat{h}^n \rangle = (g-1)^{-(1+n)/2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^1 (\hat{b}_{\mathrm{s}}^{\dagger})^n|\mathrm{vac}\rangle - ng(g-1)^{-(1+n)/2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^0 (\hat{b}_{\mathrm{s}}^{\dagger})^{n-1}|\mathrm{vac}\rangle, \tag{7}$$

$$\langle (\hat{h}^{\dagger})^m \hat{h}^1 \rangle = (g-1)^{-(m+1)/2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^m (\hat{b}_{\mathrm{s}}^{\dagger})^1|\mathrm{vac}\rangle - mg(g-1)^{-(m+1)/2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^{m-1} (\hat{b}_{\mathrm{s}}^{\dagger})^0|\mathrm{vac}\rangle. \tag{8}$$

Finally, if moments up to the fourth are of interest, i.e. $n+m \leq 4$ as, e.g., in [35], then we add the missing moment

$$\langle (\hat{h}^{\dagger})^2 \hat{h}^2 \rangle = (g-1)^{-2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^2 (\hat{b}_{\mathrm{s}}^{\dagger})^2|\mathrm{vac}\rangle - 4g(g-1)^{-2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^1 (\hat{b}_{\mathrm{s}}^{\dagger})^1|\mathrm{vac}\rangle + 2g^2(g-1)^{-2} \langle \mathrm{vac}|\hat{b}_{\mathrm{s}}^0 (\hat{b}_{\mathrm{s}}^{\dagger})^0|\mathrm{vac}\rangle. \tag{9}$$

Once normally ordered moments of noise mode are calculated, the antinormally ordered moments of the microwave radiation mode are found from experimental data by solving the linear system (3) with respect to $\langle \hat{a}^i (\hat{a}^{\dagger})^j \rangle$. In the same way, the choice $\hat{b}=\hat{b}_{\mathrm{i}}$ enables calculation of the antinormally ordered moments of noise mode and normally ordered moments of the original microwave mode. Relations between differently ordered moments are given in Ref. [50].

Since ordered moments are extracted from experimental data, it is reasonable to compare them with theoretical values. Let us consider examples of quantum one-mode states and calculate the corresponding normally and antinormally ordered moments.

As far as Fock states $|N\rangle$ are concerned, we have

$$\langle N|(\hat{a}^{\dagger})^n \hat{a}^m|N\rangle = \begin{cases} \dfrac{N!}{(N-n)!} & \text{if } n=m\leq N, \\ 0 & \text{otherwise;} \end{cases} \tag{10}$$

$$\langle N|\hat{a}^k (\hat{a}^{\dagger})^l|N\rangle = \frac{(N+k)!}{N!}\delta_{kl}. \tag{11}$$

Among mixed states, we point out a thermal state given by the unitless temperature T. Moments are obviously related with the average photon number and read

$$\langle (\hat{a}^{\dagger})^n \hat{a}^m \rangle_{\mathrm{thermal}} = \frac{n!\delta_{nm}}{\left(e^{1/T}-1\right)^n}, \tag{12}$$

$$\langle \hat{a}^k (\hat{a}^{\dagger})^l \rangle_{\mathrm{thermal}} = \frac{k!\delta_{kl}}{\left(1-e^{-1/T}\right)^k}. \tag{13}$$

Classical-like states are represented by a family of coherent states $|\alpha\rangle$, i.e. eigenstates of the annihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. These states can be generated, e.g., by masers, and represent an electromagnetic field of a local oscillator if $|\alpha| \gg 1$. The ordered moments are

$$\langle\alpha|(\hat{a}^\dagger)^n\hat{a}^m|\alpha\rangle = (\alpha^*)^n\alpha^m, \tag{14}$$

$$\langle\alpha|\hat{a}^k(\hat{a}^\dagger)^l|\alpha\rangle = \sum_{p=0}^{\min(k,l)} \begin{pmatrix} k \\ p \end{pmatrix} \begin{pmatrix} l \\ p \end{pmatrix} p!\alpha^{k-p}(\alpha^*)^{l-p}. \tag{15}$$

Other examples of nonclassical pure states of light are even/odd coherent states $|\pm\rangle = \mathcal{N}_\pm(|\alpha\rangle \pm |-\alpha\rangle)$ with $\mathcal{N}_\pm = \left[2\left(1 \pm \exp(-2|\alpha|^2)\right)\right]^{-1/2}$, which were introduced in Ref. [43]. Transformation of these states after passing a linear amplifier is considered in Ref. [51]. Here, we calculate all the ordered moments and then explore their non-unitary evolution in Sec. VII. We have

$$\langle\pm|(\hat{a}^\dagger)^n\hat{a}^m|\pm\rangle = \mathcal{N}_\pm^2(\alpha^*)^n\alpha^m \times\left[1 + (-1)^{n+m} \pm e^{-2|\alpha|^2}\left((-1)^n + (-1)^m\right)\right], \tag{16}$$

$$\langle\pm|\hat{a}^k(\hat{a}^\dagger)^l|\pm\rangle = \mathcal{N}_\pm^2 \sum_{p=0}^{\min(k,l)} \begin{pmatrix} k \\ p \end{pmatrix}\begin{pmatrix} l \\ p \end{pmatrix} p!\ \alpha^{k-p}(\alpha^*)^{l-p} \times\left[1+(-1)^{k+l-2p} \pm e^{-2|\alpha|^2}\left((-1)^{k-p}+(-1)^{l-p}\right)\right]. \tag{17}$$

III. TOMOGRAM

As it is stated in Sec. I, apart from the density operator $\hat{\rho}$, the state of one-mode photon state can alternatively be described by the symplectic tomogram [12]

$$w_{\rm s}(X,\mu,\nu) = {\rm Tr}\left[\hat{\rho}\delta(X - \mu\hat{q} - \nu\hat{p})\right], \tag{18}$$

where X is the real argument, μ and ν are real parameters, $\hat{q}$ and $\hat{p}$ are quadrature operators. In what follows, we will be interested in the measurable characteristic, viz., the so-called 'optical' tomogram $w(X,\theta) = w_{\rm s}(X,\mu=\cos\theta,\nu=\sin\theta)$, where $\theta \in [0,2\pi]$ is the phase of the local oscillator. Here, we use the adjective 'optical' as a conventional one, although microwave quantum states are under study. Unless otherwise stated, we will refer to the 'optical' tomogram as tomogram simply.

Analysis of the microwave quantum states employs linear amplifiers, so the tomogram $w_{\rm amp}(X,\theta)$ of the amplified signal is only accessible. Our goal is to obtain the relation between the tomogram of amplified quantum state and the tomogram of the original one. As shown in Ref. [52], the amplification (1) is represented by a convolution of the signal and idler fields in phase space. The Wigner function of the amplified signal reads [52]

$$W_{\rm amp}(\alpha) = \frac{1}{g}\int d^2\beta\ W\left(\frac{\alpha - \sqrt{g-1}\beta}{\sqrt{g}}\right) W_{\rm noise}(\beta), \tag{19}$$

where $\alpha = (q+ip)/\sqrt{2}$. Assuming $g \gg 1$, Eq. (19) can be readily simplified. Integrating such a simplified Wigner function with delta-function $\delta(X - q\cos\theta - p\sin\theta)$ yields the convolution expression for tomograms

$$w_{\rm amp}(X,\theta) = \frac{1}{\sqrt{g}}\int d^2\beta\ w\left(\frac{X}{\sqrt{g}} - \frac{\beta e^{i\theta} + \beta^* e^{-i\theta}}{\sqrt{2}}, \theta\right) \times W_{\rm noise}(\beta). \tag{20}$$

Associating $w_{\rm amp}(X,\theta)$ with measured data, it is possible, in general, to get insight into the state tomogram $w(X,\theta)$ itself by fulfilling the deconvolution of (20), provided the thorough knowledge of the noise state. The noise mode is usually assumed to be in thermal state with an experimentally controllable temperature T [34]. If this is the case, then

$$w_{\rm amp}(X,\theta) = \frac{1}{\sqrt{2\pi\sigma^2 g}}\int_{-\infty}^{+\infty} w\left(\frac{X}{\sqrt{g}} - Y, \theta\right) \exp\left[-\frac{Y^2}{2\sigma^2}\right] dY, \tag{21}$$

where Gaussian variance σ is expressed in terms of the unitless temperature T via formula $\sigma = \sqrt{\frac{1}{2}\coth\frac{1}{2T}}$.

While processing the experimental data, it is convenient to deal with tomographic moments $\langle X_\theta^r\rangle = \int_{-\infty}^{+\infty} X^r w(X,\theta) d\theta$. These moments are known to contain the full information about the quantum state. Using (21), we get the connection between tomographic moments of the amplified signal $\langle X_\theta^r\rangle_{\rm amp}$ and tomographic moments of the original microwave photon state $\langle X_\theta^i\rangle$:

$$\langle X_\theta^r\rangle_{\rm amp} = \sqrt{g^r}\sum_{l=0}^{[r/2]} \begin{pmatrix} r \\ 2l \end{pmatrix} (2l-1)!!\langle X_\theta^{r-2l}\rangle\sigma^{2l}, \tag{22}$$

where $(2l-1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2l-1)$ and $(-1)!! = 1$. All the tomographic moments $\langle X_\theta^i\rangle$ up to a desired order R can be readily obtained from experimentally accessible tomographic moments $\langle X_\theta^r\rangle_{\rm amp}$, $r = 0,1,\ldots,R$, by inversion of formula (22). After that, the tomographic probability-distribution function can be also expressed through tomographic moments.

IV. RELATION BETWEEN TOMOGRAMS AND MOMENTS

Let us now derive relations between measurable moments and measurable tomographic distributions. These relations are to be used as a cross check of two approaches to probing microwave quantum states.

Using the formalism of characteristic functions $\langle e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}\rangle$ and $\langle e^{-\lambda^*\hat{a}}e^{\lambda\hat{a}^\dagger}\rangle$, it is not hard to prove that the Wigner function is expressed through normally-

ordered and antinormally-ordered moments as follows:

$$W(\alpha) = \sum_{n,m} \frac{\langle (\hat{a}^\dagger)^n \hat{a}^m \rangle}{\pi^2 n! m!} \int d^2\lambda \ \lambda^n (-\lambda^*)^m \times \exp\left[-\frac{1}{2}|\lambda|^2 + \alpha\lambda^* - \alpha^*\lambda \right], \quad (23)$$

$$W(\alpha) = \int d^2\lambda \left(\sum_{k,l} \frac{\langle \hat{a}^k (\hat{a}^\dagger)^l \rangle (-\lambda^*)^k \lambda^l}{\pi^2 k! l!} \right) \times \exp\left[\frac{1}{2}|\lambda|^2 + \alpha\lambda^* - \alpha^*\lambda \right]. \quad (24)$$

It is worth noting that the expression (23) implies integration over complex plane and then summation on $n, m = 0, 1, \ldots$, whereas formula (24) avoids singularities if the summation is done before the integration. As it is shown later, this peculiarity leads to simpler expressions for normally ordered moments than for antinormally ordered ones. In Ref. [53] a discussion is presented how to reveal negativity of the Wigner function (23) by the analysis of higher-order moments.

Our goal is the relation between the 'optical' tomogram and the moments but at first we exploit the known mapping of the Wigner function onto symplectic tomogram (see, e.g., [12]) and obtain after simplification

$$w(X,\mu,\nu) = \frac{1}{\sqrt{\pi}} \exp\left[-\frac{X^2}{\mu^2+\nu^2} \right] \sum_{n,m} \frac{\langle (\hat{a}^\dagger)^n \hat{a}^m \rangle}{n! m!} \times \frac{(\mu+i\nu)^n (\mu-i\nu)^m}{\sqrt{2^{n+m}(\mu^2+\nu^2)^{(n+m+1)}}} \ H_{n+m}\left(\frac{X}{\sqrt{\mu^2+\nu^2}} \right), \quad (25)$$

$$w(X,\mu,\nu) = \frac{1}{2\pi} \int d\xi \ \exp\left[\frac{\xi^2}{4}(\mu^2+\nu^2) - i\xi X \right] \times \left(\sum_{k,l} \frac{\langle \hat{a}^k (\hat{a}^\dagger)^l \rangle}{k! l!} \left(\frac{i\xi}{\sqrt{2}} \right)^{k+l} (\mu-i\nu)^k (\mu+i\nu)^l \right), \quad (26)$$

where $H_N(X)$ is the Hermite polynomial of degree N. Substituting $\cos\theta$ for μ and $\sin\theta$ for ν in (25)–(26), the 'optical' tomogram $w(X,\theta)$ is expressed in terms of ordered moments

$$w(X,\theta) = \frac{e^{-X^2}}{\sqrt{\pi}} \sum_{n,m} \frac{\langle (\hat{a}^\dagger)^n \hat{a}^m \rangle e^{i(n-m)\theta}}{\sqrt{2^{n+m}} n! m!} \ H_{n+m}(X) \quad (27)$$

$$= \int \frac{d\xi}{2\pi} \ e^{\xi^2/4 - i\xi X} \left(\sum_{k,l} \frac{\langle \hat{a}^k (\hat{a}^\dagger)^l \rangle}{k! l!} \left(\tfrac{i\xi}{\sqrt{2}} \right)^{k+l} e^{i(l-k)\theta} \right). \quad (28)$$

Since both tomogram and moments are measurable in microwave experiments, Eq. (27) is the basis for a cross-check of homodyne and heterodyne detection schemes.

Let us now address the problem to invert formulas (27)–(28). Using orthogonality of Hermite polynomials and orthogonality of trigonometric functions, we have the following inverse relation:

$$\langle (\hat{a}^\dagger)^n \hat{a}^m \rangle = \frac{n! m!}{2\pi \sqrt{2^{n+m}} (n+m)!} \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dX \ w(X,\theta) \times e^{i(m-n)\theta} H_{n+m}(X) \quad (29)$$

$$= \frac{n! m!}{2\pi \sqrt{2^{n+m}} (n+m)!} \int_0^{2\pi} d\theta \ e^{i(m-n)\theta} \langle H_{n+m}(X_\theta) \rangle, \quad (30)$$

where $\langle H_{n+m}(X_\theta) \rangle$ depends on θ and is obtained from polynomial $H_{n+m}(X)$ by replacing $X^r \to \langle X_\theta^r \rangle$ for all $r = 0, 1, \ldots, n+m$. Thus, formula (30) establishes a relation between normally ordered moments and tomographic moments. We must note that the relation (29) was found previously in the papers [38, 39].

As far as antinormally ordered moments are concerned, a direct use of Eq. (28) is sophisticated so we resort to the connection between antinormally and normally ordered moments and exploit result of Eqs. (29)–(30) to obtain

$$\langle \hat{a}^k (\hat{a}^\dagger)^l \rangle = \frac{k! l!}{2\pi \sqrt{2^{k+l}}} \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dX \ w(X,\theta) \ e^{i(k-l)\theta} \times \sum_{p=0}^{\min(k,l)} \frac{2^p H_{k+l-2p}(X)}{p!(k+l-2p)!} \quad (31)$$

$$= \frac{k! l!}{2\pi \sqrt{2^{k+l}}} \int_0^{2\pi} d\theta \ e^{i(k-l)\theta} \sum_{p=0}^{\min(k,l)} \frac{2^p \langle H_{k+l-2p}(X_\theta) \rangle}{p!(k+l-2p)!}. \quad (32)$$

In order to perform cross check of the experimental data (tomogram vs. moments), one can also compare measured tomographic moments $\langle X_\theta^r \rangle$ with tomographic moments predicted by the measurement of normally ordered moments $\langle (\hat{a}^\dagger)^n \hat{a}^m \rangle$, namely,

$$\langle X_\theta^r \rangle = \sum_{\substack{n+m \le r \\ r-n-m \text{ is even}}} \frac{r! \sqrt{2^{n+m-2r}}}{n! m! \left(\frac{r-n-m}{2} \right)!} \langle (\hat{a}^\dagger)^n \hat{a}^m \rangle e^{i(n-m)\theta}. \quad (33)$$

The latter formula is obtained by combining a definition of the tomographic moment $\langle X_\theta^r \rangle$, Eq. (27), and the following integral

$$\int_{-\infty}^{+\infty} X^r H_N(X) e^{-X^2} dX = \begin{cases} 0 & \text{if } r-N \text{ is odd or } N > r, \\ \dfrac{\sqrt{\pi} r!}{2^{r-N} \left(\frac{r-N}{2} \right)!} & \text{otherwise,} \end{cases} \quad (34)$$

which can also be written in the form of two-dimensional Hermite polynomial [54].

A. Purity in terms of moments

In this subsection, we attack the problem how to determine purity of the state without unnecessary reconstruction of the density operator but dealing with measured normally ordered moments. Just as the measurable tomogram is used for fast and reliable calculation of the purity parameter [15], here we derive the purity in terms of moments only.

To start with, the overlap $\mathrm{Tr}[\hat{\rho}_1\hat{\rho}_2]$ between two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ equals to the overlap of the corresponding Wigner functions. In view of the relation (23), it is readily seen that

$$\mathrm{Tr}[\hat{\rho}_1\hat{\rho}_2] = \sum_{n,m,k,l} \frac{(-1)^{m+k}(n+k)!}{n!m!k!l!}\delta_{n+k,m+l} \times\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle_1\langle(\hat{a}^\dagger)^k\hat{a}^l\rangle_2. \quad (35)$$

For example, if one is interested in how close the prepared state is to the vacuum one, it is enough to calculate $\langle 0|\hat{\rho}|0\rangle = \sum_k(-1)^k(k!)^{-1}\langle(\hat{a}^\dagger)^k\hat{a}^k\rangle$.

Finally, the state is thoroughly described by moments $\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle$ and its purity $\tilde{\pi}$ is

$$\tilde{\pi} = \sum_{n,m,k,l} \frac{(-1)^{m+k}(n+k)!}{n!m!k!l!}\ \delta_{n+k,m+l}\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle\langle(\hat{a}^\dagger)^k\hat{a}^l\rangle. \quad (36)$$

Purity of the thermal state (12) is $\tilde{\pi}_{\mathrm{thermal}} = \tanh(1/2T)$ so the effective temperature T_{eff} of the electromagnetic field can be probed directly via measured moments as $T_{\mathrm{eff}} = 1/(2\ \mathrm{arctanh}\tilde{\pi})$.

V. UNCERTAINTY RELATIONS

The role of uncertainty relations in quantum mechanics can scarcely be overestimated. Therefore, a direct probing of uncertainty relations in experiments is of great interest. The information in amplified microwave signals is obscured by the noise. However, as it is shown in Sec. II, the ordered moments of the original microwave mode can still be extracted from the data. The question arises itself whether such extracted moments satisfy the uncertainty relations. The violation of these inequalities would indicate the incorrectness of data processing or measurement of the data.

Let us start with a conventional Schrödinger-Robertson inequality $\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq 1/4$. Written in the form of tomographic moments, i.e. $\langle(\Delta X_\theta)^2\rangle = \langle X_\theta^2\rangle - \langle X_\theta\rangle^2$, this inequality takes the form [55]

$$\langle(\Delta X_\theta)^2\rangle\langle(\Delta X_{\theta+\pi/2})^2\rangle - \Big[\langle(\Delta X_{\theta+\pi/4})^2\rangle - \frac{1}{2}\Big(\langle(\Delta X_\theta)^2\rangle + \langle(\Delta X_{\theta+\pi/2})^2\rangle\Big)\Big]^2 \geq \frac{1}{4} \quad (37)$$

and is to be satisfied for any angle $\theta \in [0, 2\pi]$. Using the established relation between tomographic moments and normally ordered moments (33), we have $\langle X_\theta\rangle = \left(\langle\hat{a}^\dagger\rangle e^{i\theta} + \langle\hat{a}\rangle e^{-i\theta}\right)/\sqrt{2}$ and $\langle X_\theta^2\rangle = 1/2 + \langle\hat{a}^\dagger\hat{a}\rangle + \left(\langle(\hat{a}^\dagger)^2\rangle e^{i2\theta} + \langle\hat{a}^2\rangle e^{-i2\theta}\right)/2$. Finally, we obtain the uncertainty relation in terms of moments

$$\big(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle\big) + \big(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle\big)^2 - \big(\langle(\hat{a}^\dagger)^2\rangle - \langle\hat{a}^\dagger\rangle^2\big)\big(\langle\hat{a}^2\rangle - \langle\hat{a}\rangle^2\big) \ \geq\ 0, \quad (38)$$

which turns out to be independent on θ.

The inequality (38) can be made stronger if we take into account the purity of the state. Indeed, we have [13] $\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq \Phi^2(\tilde{\pi})/4$, where $\Phi(\tilde{\pi}) \approx (4 + \sqrt{16 + 9\tilde{\pi}^2})/9\tilde{\pi}$ within the accuracy of 1%. Consequently, purity-dependent uncertainty relation in terms of normally ordered moments is

$$\big(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle\big) + \big(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle\big)^2 - \big(\langle(\hat{a}^\dagger)^2\rangle - \langle\hat{a}^\dagger\rangle^2\big)\big(\langle\hat{a}^2\rangle - \langle\hat{a}\rangle^2\big) \geq \big(\Phi^2(\tilde{\pi}) - 1\big)/4, \quad (39)$$

where the purity $\tilde{\pi}$ is to be calculated via moments in accordance with Eq. (36). Thus, we have formulated a self-consistent problem of checking purity-dependent uncertainty relations by using measurable moments only.

Another drawback of inequality (38) is that it exploits the lowest moments only, i.e. $\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle$ with $n + m \leq 2$. Let us derive uncertainty relations in terms of moments such that they involve moments up to a desired order. In fact, suppose the operator $\hat{F} = z_0\mathbb{1} + \sum_n(y_n\hat{a}^n + z_n(\hat{a}^\dagger)^n)$ with arbitrary complex numbers z_0, y_n, z_n. Then the mean value $\langle\hat{F}^\dagger\hat{F}\rangle \geq 0$ for all $z_0, y_n, z_n \in \mathbb{C}$. This immediately implies that the quadratic form is positive-semidefinite. According to Sylvester's criterion, all leading principal minors of the corresponding matrix are nonnegative. For instance, we readily obtain the inequalities on moments up to the fourth order:

$$\begin{pmatrix} \langle\mathbb{1}\rangle & \langle\hat{a}\rangle & \langle\hat{a}^\dagger\rangle & \langle\hat{a}^2\rangle & \langle(\hat{a}^\dagger)^2\rangle \\ \langle\hat{a}^\dagger\rangle & \langle\hat{a}^\dagger\hat{a}\rangle & \langle(\hat{a}^\dagger)^2\rangle & \langle\hat{a}^\dagger\hat{a}^2\rangle & \langle(\hat{a}^\dagger)^3\rangle \\ \langle\hat{a}\rangle & \langle\hat{a}^2\rangle & \langle\hat{a}\hat{a}^\dagger\rangle & \langle\hat{a}^3\rangle & \langle\hat{a}(\hat{a}^\dagger)^2\rangle \\ \langle(\hat{a}^\dagger)^2\rangle & \langle(\hat{a}^\dagger)^2\hat{a}\rangle & \langle(\hat{a}^\dagger)^3\rangle & \langle(\hat{a}^\dagger)^2\hat{a}^2\rangle & \langle(\hat{a}^\dagger)^4\rangle \\ \langle\hat{a}^2\rangle & \langle\hat{a}^3\rangle & \langle\hat{a}^2\hat{a}^\dagger\rangle & \langle\hat{a}^4\rangle & \langle\hat{a}^2(\hat{a}^\dagger)^2\rangle \end{pmatrix} \geq 0. \quad (40)$$

If normally ordered moments are measured, then one should replace $\langle\hat{a}\hat{a}^\dagger\rangle = \langle\hat{a}^\dagger\hat{a}\rangle + 1$, $\langle\hat{a}^2\hat{a}^\dagger\rangle = \langle\hat{a}^\dagger\hat{a}^2\rangle + 2\langle\hat{a}\rangle$, $\langle\hat{a}(\hat{a}^\dagger)^2\rangle = \langle(\hat{a}^\dagger)^2\hat{a}\rangle + 2\langle\hat{a}^\dagger\rangle$, $\langle\hat{a}^2(\hat{a}^\dagger)^2\rangle = \langle(\hat{a}^\dagger)^2\hat{a}^2\rangle + 4\langle\hat{a}^\dagger\hat{a}\rangle + 2$. Otherwise, i.e. if antinormally ordered moments are experimentally determined, then one should replace $\langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{a}\hat{a}^\dagger\rangle - 1$, $\langle\hat{a}^\dagger\hat{a}^2\rangle = \langle\hat{a}^2\hat{a}^\dagger\rangle - 2\langle\hat{a}\rangle$, $\langle(\hat{a}^\dagger)^2\hat{a}\rangle = \langle\hat{a}(\hat{a}^\dagger)^2\rangle - 2\langle\hat{a}^\dagger\rangle$, $\langle(\hat{a}^\dagger)^2\hat{a}^2\rangle = \langle\hat{a}^2(\hat{a}^\dagger)^2\rangle - 4\langle\hat{a}\hat{a}^\dagger\rangle + 2$. Note that formula (38) is nothing else but a condition on nonnegativity of the second principal minor of the matrix (40).

VI. UNITARY EVOLUTION AND EIGENSTATES

In this section, we follow ideas of the seminal paper [41]. Normally and antinormally ordered moments are functions of non-commuting operators $\hat{a}$ and $\hat{a}^\dagger$ and can be considered as specific 'phase-space quasidistributions'. Now we are aimed at deriving the laws which govern the transformation with time of these 'phase-space quasidistributions'. In other words, the problem is to find time evolution equations for ordered moments. If this problem is solved, then such equations can be used as an alternative to the Schrödinger equation for the wave function, the von Neumann equation for the density operator, and the Moyal equation for the Wigner function. The crucial point is that ordered moments are experimentally measurable characteristics of microwave quantum states in contrast to wave functions, density operators, and Wigner functions.

We start with correspondence rules between the operators acting on the Wigner function and the operators acting on the ordered moments. Applying operators q, p, $\partial/\partial q$, and $\partial/\partial p$ to the left-hand side of Eq. (23), we obtain

$$\frac{\partial}{\partial q} \leftrightarrow -\tfrac{1}{\sqrt{2}}\big(n\hat{\Delta}^{(\mathrm{n})}_{-1,0} + m\hat{\Delta}^{(\mathrm{n})}_{0,-1}\big), \tag{41}$$

$$\frac{\partial}{\partial p} \leftrightarrow \tfrac{i}{\sqrt{2}}\big(n\hat{\Delta}^{(\mathrm{n})}_{-1,0} - m\hat{\Delta}^{(\mathrm{n})}_{0,-1}\big), \tag{42}$$

$$q \leftrightarrow \tfrac{1}{\sqrt{2}}\big(\hat{\Delta}^{(\mathrm{n})}_{+1,0}+\hat{\Delta}^{(\mathrm{n})}_{0,+1}\big)+\tfrac{1}{2\sqrt{2}}\big(n\hat{\Delta}^{(\mathrm{n})}_{-1,0}+m\hat{\Delta}^{(\mathrm{n})}_{0,-1}\big), \tag{43}$$

$$p \leftrightarrow \tfrac{i}{\sqrt{2}}\big(\hat{\Delta}^{(\mathrm{n})}_{+1,0}-\hat{\Delta}^{(\mathrm{n})}_{0,+1}\big)-\tfrac{i}{2\sqrt{2}}\big(n\hat{\Delta}^{(\mathrm{n})}_{-1,0}-m\hat{\Delta}^{(\mathrm{n})}_{0,-1}\big), \tag{44}$$

where a displacement operator $\hat{\Delta}^{(\mathrm{n})}_{i,j}$ for the normally ordered moments is introduced as follows:

$$\hat{\Delta}^{(\mathrm{n})}_{i,j}\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle := \langle(\hat{a}^\dagger)^{n+i}\hat{a}^{m+j}\rangle. \tag{45}$$

Arguing as above and employing Eq. (24), the correspondence relations for antinormally ordered moments are found

$$\frac{\partial}{\partial q} \leftrightarrow -\tfrac{1}{\sqrt{2}}\big(k\hat{\Delta}^{(\mathrm{a})}_{-1,0} + l\hat{\Delta}^{(\mathrm{a})}_{0,-1}\big), \tag{46}$$

$$\frac{\partial}{\partial p} \leftrightarrow -\tfrac{i}{\sqrt{2}}\big(k\hat{\Delta}^{(\mathrm{a})}_{-1,0} - l\hat{\Delta}^{(\mathrm{a})}_{0,-1}\big), \tag{47}$$

$$q \leftrightarrow \tfrac{1}{\sqrt{2}}\big(\hat{\Delta}^{(\mathrm{a})}_{+1,0}+\hat{\Delta}^{(\mathrm{a})}_{0,+1}\big)-\tfrac{1}{2\sqrt{2}}\big(k\hat{\Delta}^{(\mathrm{a})}_{-1,0}+l\hat{\Delta}^{(\mathrm{a})}_{0,-1}\big), \tag{48}$$

$$p \leftrightarrow -\tfrac{i}{\sqrt{2}}\big(\hat{\Delta}^{(\mathrm{a})}_{+1,0}-\hat{\Delta}^{(\mathrm{a})}_{0,+1}\big)-\tfrac{i}{2\sqrt{2}}\big(k\hat{\Delta}^{(\mathrm{a})}_{-1,0}-l\hat{\Delta}^{(\mathrm{a})}_{0,-1}\big), \tag{49}$$

with a displacement operator $\hat{\Delta}^{(\mathrm{a})}_{i,j}$ for the antinormally ordered moments being

$$\hat{\Delta}^{(\mathrm{a})}_{i,j}\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle := \langle\hat{a}^{k+i}(\hat{a}^\dagger)^{l+j}\rangle. \tag{50}$$

The time evolution equation for ordered moments is readily obtained from the Moyal equation

$$\left[\frac{\partial}{\partial t}+p\frac{\partial}{\partial q}-\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)!4^r}\frac{d^{2r+1}U}{dq^{2r+1}}\frac{\partial^{2r+1}}{\partial p^{2r+1}}\right]W=0 \tag{51}$$

by substituting normally or antinormally ordered moments for W and using the correspondence table (41)–(44) or (46)–(49), respectively.

The stationary Schrödinger equation, i.e. the eigenstate problem $\hat{H}|E\rangle = E|E\rangle$, transforms into the known equation for the Wigner function

$$\left[\frac{p^2}{2}-\frac{1}{8}\frac{\partial^2}{\partial q^2}+\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!4^r}\frac{d^{2r}U}{dq^{2r}}\frac{\partial^{2r}}{\partial p^{2r}}\right]W_E = EW_E \tag{52}$$

and then to the equation for ordered moments according to associations (41)–(44) and (46)–(49).

In order to demonstrate the equations for moments, we consider the simplest case of the free evolution of the electromagnetic field, which is effectively governed by the harmonic oscillator potential $U = q^2/2$. If this is the case, the lowest-order derivatives are only presented in Eqs. (51) and (52). For normally ordered moments we have

$$\left[\frac{\partial}{\partial t} - i(n-m)\right]\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle = 0, \tag{53}$$

$$\left[\hat{\Delta}^{(\mathrm{n})}_{+1,+1} + \tfrac{n+m+1}{2}\right]\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle_E = E\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle_E. \tag{54}$$

It is not hard to see that the moments (10) multiplied by $e^{i(n-m)t}$ are solutions of both Eqs. (53) and (54) if $E = N + 1/2$.

Likewise, the antinormally ordered moments (11) multiplied by $e^{i(l-k)t}$ are solutions of equations

$$\left[\frac{\partial}{\partial t} + i(k-l)\right]\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle = 0, \tag{55}$$

$$\left[\hat{\Delta}^{(\mathrm{a})}_{+1,+1} - \tfrac{k+l+1}{2}\right]\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle_E = E\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle_E. \tag{56}$$

VII. DAMPED EVOLUTION

In practice, microwave transmission lines have loss due to a finite conductivity of waveguide walls or lossy dielectric [31]. In many experiments the loss may be neglected. On the other hand, quantum superpositions are very vulnerable to the relaxation and decoherence while interacting with the environment. In view of this, the effect of loss is of interest.

To describe a microwave quantum state in lossy environment we make use of a damped quantum oscillator model [42, 56]. The Wigner function obeys the linear Fokker-Planck equation of the form [56]

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+p\frac{\partial}{\partial q}-q\frac{\partial}{\partial p}\right]W \\ &= \left[2\gamma\frac{\partial}{\partial p}p+\frac{\gamma}{2\omega}\left(\frac{\partial^2}{\partial q^2}+\frac{\partial^2}{\partial p^2}\right)-\frac{\gamma^2}{\omega}\frac{\partial^2}{\partial p\partial q}\right]W,\end{aligned} \tag{57}$$

where γ is the damping coefficient and $\omega = \sqrt{1-\gamma^2}$. In view of correspondence relations (41)–(44), Eq. (57)

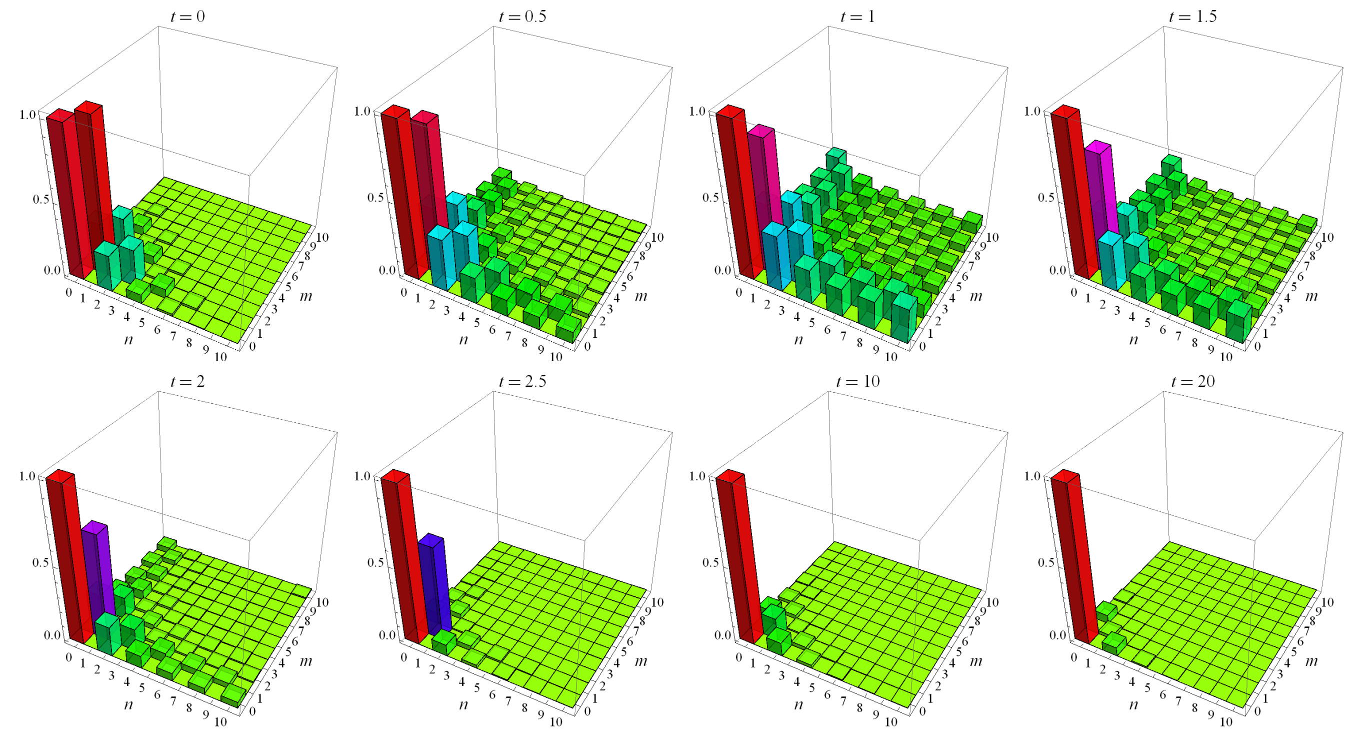

FIG. 1: (Color online) Snapshots of the normally ordered moments $|\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle|$ of the odd coherent state (16) with $\alpha = 0.5$ at eight successive times of damped evolution (58) with $\gamma = 0.1$.

transforms into the following equation for measurable normally ordered moments:

$$\left[\frac{\partial}{\partial t} - i(n-m)\right]\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle = -\gamma\,\Big[\,n+m \\ -n\hat{\Delta}^{(\mathrm{n})}_{-1,+1} - m\hat{\Delta}^{(\mathrm{n})}_{+1,-1} - (\omega^{-1}-1)nm\hat{\Delta}^{(\mathrm{n})}_{-1,-1} \\ -\tfrac{1}{2}(1+i\omega^{-1}\gamma)n(n-1)\hat{\Delta}^{(\mathrm{n})}_{-2,0} \\ -\tfrac{1}{2}(1-i\omega^{-1}\gamma)m(m-1)\hat{\Delta}^{(\mathrm{n})}_{0,-2}\Big]\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle. \qquad (58)$$

In the same way, one can construct a time evolution equation for antinormally ordered moments.

The infinite system of equations (58) is linear. Thus, the partial differential equation (57) is reduced to a linear system of the first order differential equations. This means that a formal solution of the system is the matrix exponent which is easy to compute if one is interested in time evolution of the lower-order moments. Moreover, any linear dynamics of the system (not necessarily quadratic as in Eq. (57)) can be predicted numerically in terms of moments, whereas the dynamics of the Wigner function would require solving finite-difference equations with artificial mesh spacing. Conversely, moments $\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle$ are endowed by a natural 'mesh spacing' $\Delta n, \Delta m = \pm 1$ and, last but not least, are experimentally accessible.

An example of non-unitary evolution of the normally ordered moments is presented in Fig. 1. The odd coherent state $|-\rangle$ (16) with $\alpha = 0.5$ evolves in time according to the system of equations (58), where damping coefficient $\gamma = 0.1$. Observation of the moments at successive times provides snapshots of the decoherence process (cf. [57]).

VIII. CONCLUSIONS

To resume we point out the main results of our paper.

We have considered two approaches to measuring microwave quantum states, namely, the homodyne detection scheme with the 'optical' tomogram as output and the heterodyne detection scheme with output in the form of ordered moments of photon creation and annihilation operators. A microwave one-mode quantum state can be identified either with the tomogram $w(X,\theta)$; or the tomographic moments $\langle X_\theta^r\rangle$, $r = 0,1,\ldots$; or the normally ordered moments $\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle$, $n,m = 0,1,\ldots$; or the antinormally ordered moments $\langle\hat{a}^k(\hat{a}^\dagger)^l\rangle$, $k,l = 0,1,\ldots$. It was shown how to extract these quantities from measurable characteristics of the amplified microwave signal. We suggest using the established relations between the tomogram and the ordered moments (27)–(29), (31) as well as the relations between the tomographic moments and the ordered moments (30), (32), (33) as a cross check of the experimental results obtained in Ref. [32] and in Refs. [33–35].

As the normally/antinormally ordered moments are measurable and determine a quantum state, an effort to obtain new results for the ordered moments has been made. Indeed, purity is expressed through moments

and is used in purity-dependent uncertainty relation in terms of moments. Another result is that the moments are to satisfy a generalization of the inequality (40). We have obtained the time evolution equation in terms of moments, which is informationally equivalent to the von Neumann equation for the density operator and the Moyal equation for the Wigner function. The energy level problem and the non-unitary evolution of the damped microwave electromagnetic field are also considered in terms of moments. The damped evolution is described by a system of linear differential equations on moments, which is beneficial (from the viewpoint of numerical analysis) in comparison with a partial differential equation on the Wigner function.

Since normally/antinormally ordered moments are of great interest, a construction of the star-product scheme [58] for such moments is a problem for further investigation and will be considered elsewhere.

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